Inverse Kinematics Solution
Geometric Approach

* Steps for manipulators with spherical wrist (6-DOF arms)
  1- calculate the position of wrist center
  2- Based on O_c (wrist Center) Calculate q_1, q_2, q_3
  3- From q_1, q_2, q_3 Calculate ${}^0R_3$
  4- Based on the given rotation of the end effector ${}^0R_6 \rightarrow {}^3R_6 = ({}^0R_3)^{-1} {}^0R_6$
  5- Use ${}^3R_6$ to calculate $\theta_4, \theta_5, \theta_6$ based on Euler angles rotation

* Special case for $n<6$ (example: SCARA having 4-DOF)
  • The first three steps be as above.
  4- Calculate the Rotation of EE w.r.t previous frame (i.e. θ_4) from the relation $\alpha = \theta_1 + \theta_2 - \theta_4$, ($\alpha$ is the angle of EE. w.r.t base)

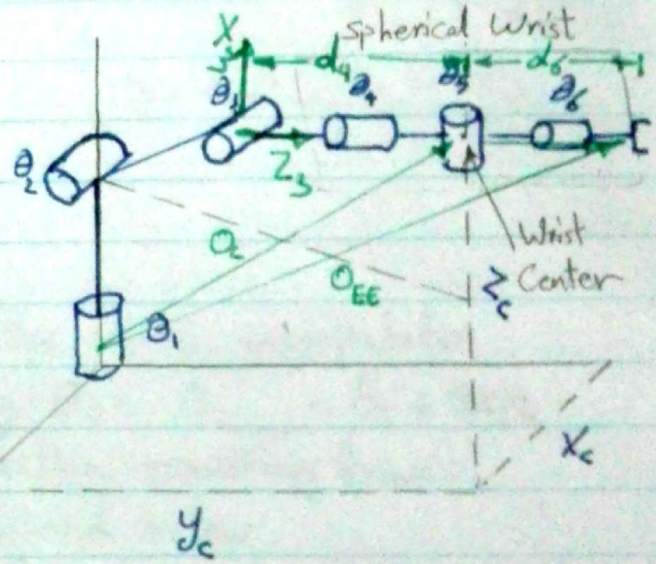
Examples

I Articulated Manipulator:-

step1: O_c Calculation

$$O_c = O_{EE} - d_6 R_{EE} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

O_{EE}: the position of EE (given)
d_6: distance in Z_6 direction, so it rotated to the base frame by R_{EE}
R_{EE}: orientation of EE (given)



step2: $O_c = [X_c, Y_c, Z_c]^T$
use Geometry of first three links to calculate θ_1, θ_2 and θ_3

We use the notation $\text{atan2}(X, Y) = \tan^{-1}\left(\frac{Y}{X}\right)$

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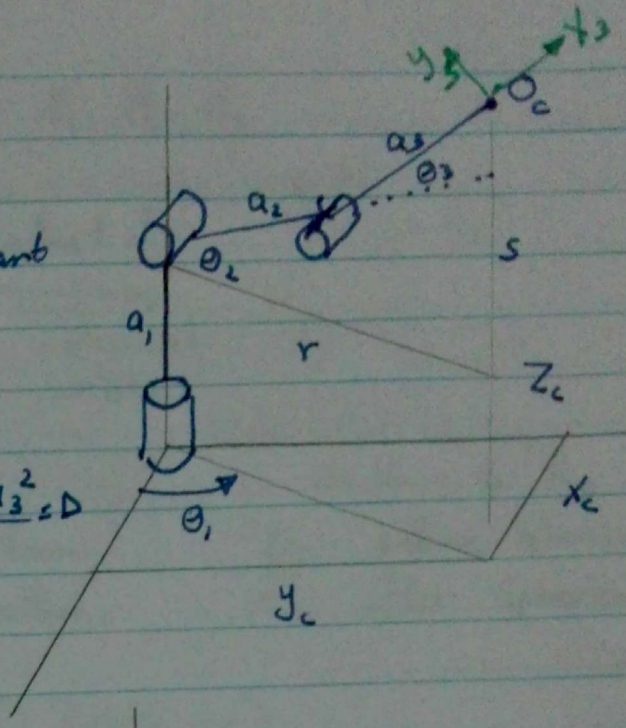
$$\theta_1 = \text{atan2}(x_c, y_c)$$

θ_2, θ_3 Calculate from the planner part of the arm

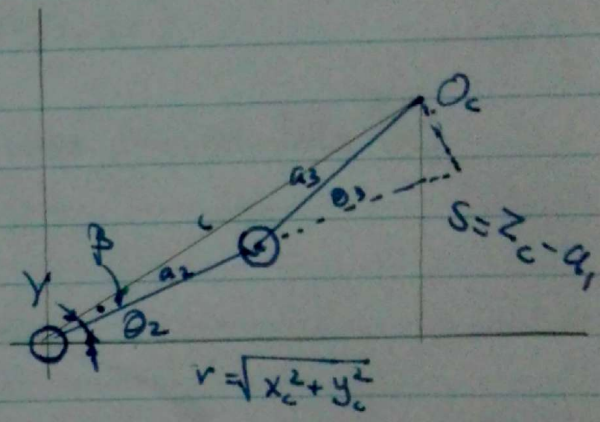


From law of Cosines:

$$\cos\theta_3 = \frac{c^2 - a_2^2 - a_3^2}{2a_2a_3} = \frac{r^2+s^2-a_2^2-a_3^2}{2a_2a_3} = D$$

$$D = \frac{(x_c^2+y_c^2)+(z_c-a_1)^2-a_2^2-a_3^2}{2a_2a_3}$$

$$\theta_3 = \text{atan2}(D, \pm\sqrt{1-D^2})$$



$$\theta_2 = \gamma - \beta$$

$$\theta_2 = \text{atan2}(\sqrt{x_c^2+y_c^2}, z_c - a_1) - \text{atan2}(a_2 + a_3\cos\theta_3, a_3\sin\theta_3)$$

$$\gamma = \tan^{-1}\frac{s}{r} = \tan^{-1}\frac{z_c - a_1}{\sqrt{x_c^2+y_c^2}}$$

$$\beta = \tan^{-1}\frac{a_3\sin\theta_3}{a_2 + a_3\cos\theta_3}$$

The above part is called (Inverse position)
The next part is called (Inverse Orientation)

step 3: Add spherical wrist part to the above manipulator. This will enforce you to modify θ_3: $\theta_{3_{new}} = \theta_3 + \pi/2$ due to the difference in x_3 direction resulting from spherical wrist, to be explained below

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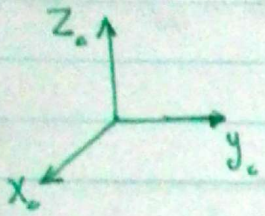
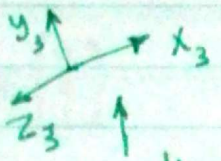
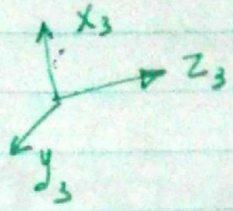
Form ${}^0R_3 = {}^0R_1\,{}^1R_2\,{}^1R_3$ based on $\theta_1, \theta_2, \theta_3$

Steps:





effect of spherical Wrist



according to $\theta_1, \theta_2, \theta_3$ calculated from Inverse position

$\therefore$ We have 2 ${}^0R_3$

${}^0R_3$ before Spherical

${}^0R_3$ after Spherical

$${}^0R_{3\ \text{after}} = {}^0R_{3\ \text{before}} * Rot(X, 90^\circ)\, Rot(Y, 90^\circ)$$

↳ the new ${}^0R_3$ is the matrix used to calculate ${}^3R_6$

step 4:

$${}^3R_6 = [{}^0R_3]^{-1}\, R_{EE}$$

Step 5:

$${}^3R_6 = \begin{bmatrix} C_4C_5C_6 - S_4S_6 & -C_4C_5S_6 - S_4C_6 & C_4S_5 \\ S_4C_5C_6 + C_4S_6 & -S_4C_5S_6 + C_4C_6 & S_4S_5 \\ -S_5C_6 & S_5S_6 & C_5 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

where $S_i = \sin(\theta_i)$, $C_i = \cos(\theta_i)$

$$\theta_4 = \text{atan2}(r_{13}, r_{23})$$
$$\theta_5 = \text{atan2}(r_{33}, \sqrt{1 - r_{33}^2})$$
$$\theta_6 = \text{atan2}(-r_{31}, r_{32})$$

Euler ZYZ angles

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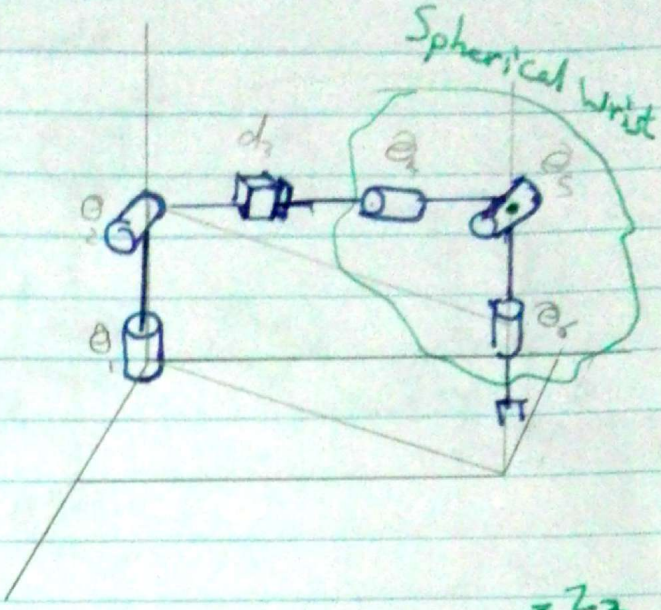
III Spherical Manipulator with Spherical Wrist



step 1:

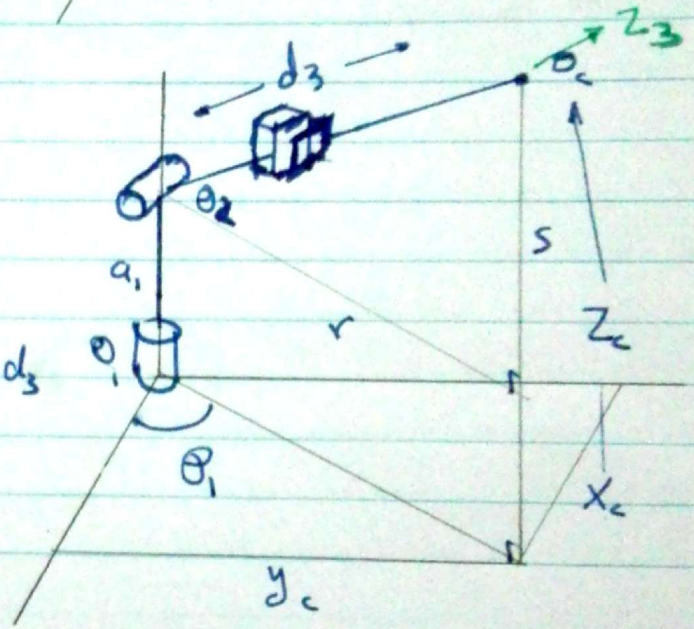
$$O_c = O_{EE} - d_6 R_{EE} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

step 2:

$$\theta_1 = \text{atan2}(x_c, y_c)$$

$$\theta_2 = \text{atan2}(\sqrt{x_c^2 + y_c^2}, z_c - a_1)$$

$$d_3 = \sqrt{x_c^2 + y_c^2 + (z_c - a_1)^2}$$



step 3:

Calculate ${}^0R_3$ based on θ_1, θ_2, d_3

step 4:

$${}^3R_6 = \left[{}^0R_3\right]^{-1} R_{EE}$$

step 5:

$${}^3R_6 = \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_4c_5s_6 - s_4c_6 & c_4s_5 \\ s_4c_5c_6 + c_4s_6 & -s_4c_5s_6 + c_4c_6 & s_4s_5 \\ -s_5c_6 & s_5s_6 & c_5 \end{bmatrix}$$

$$r = \sqrt{x_c^2 + y_c^2}$$

$$s = z_c - a_1$$

$$\theta_4 = \text{atan2}(r_{13}, r_{23})$$

$$\theta_5 = \text{atan2}(r_{23}, \sqrt{1 - r_{33}^2})$$

$$\theta_6 = \text{atan2}(-r_{31}, r_{32})$$

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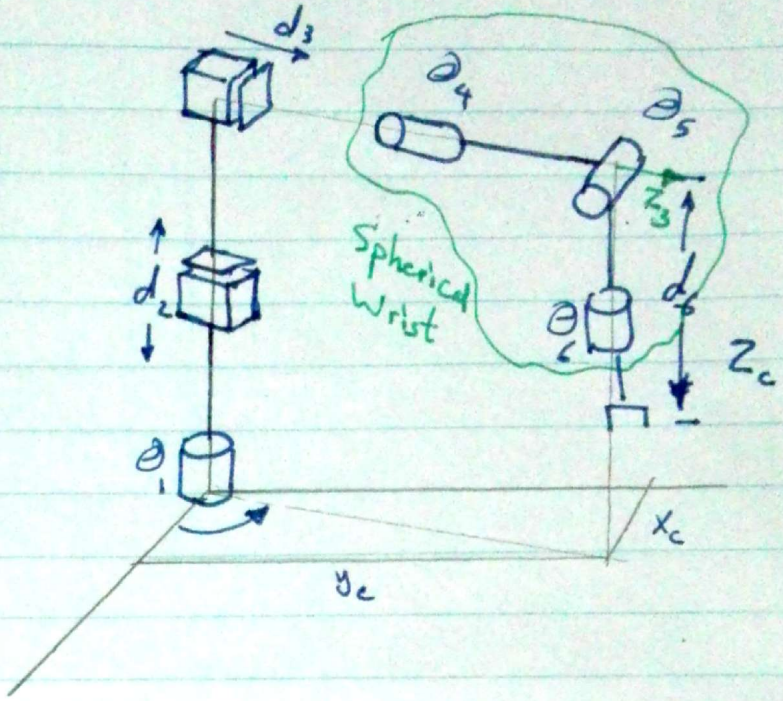
III Cylinderical Manipulator with Spherical wrist

step 1:

$$O_c = O_{EE} - d_6 R_{EE} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$



step 2:

$$\theta_1 = \text{atan2}(x_c, y_c)$$

$$d_2 = z_c$$

$$d_3 = \sqrt{x_c^2 + y_c^2}$$

Step 3:

Calculate ${}^0R_3$ based on θ_1, d_2, d_3

step 4:

$${}^3R_6 = [{}^0R_3]^{-1} \; R_{EE}$$

Step 5:

${}^3R_6$ has the form as the Previous example

$$\theta_4 = \text{atan2}(r_{13}, r_{23})$$

$$\theta_5 = \text{atan2}(r_{33}, \sqrt{1 - r_{33}^2})$$

$$\theta_6 = \text{atan2}(-r_{31}, r_{32})$$

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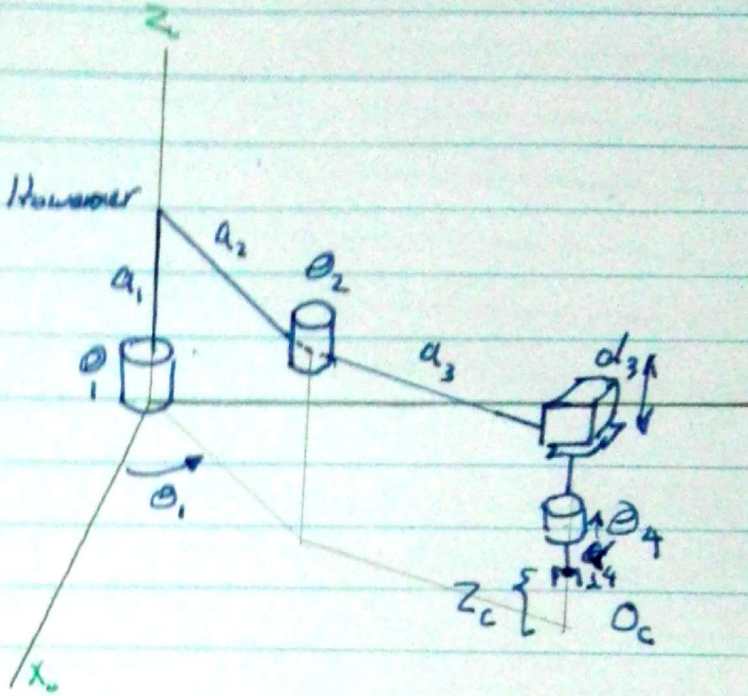
IV SCARA Robot

step 1: In this example there is no Spherical wrist found. However a virtual wrist center can be located at the End Effector position

$O_c = O_{EE}$ (given)



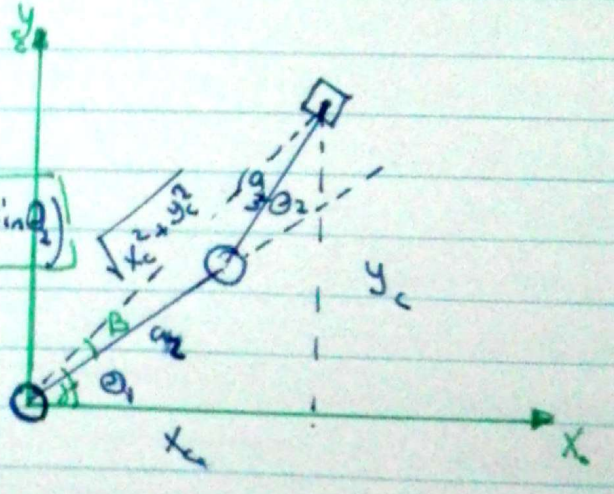
step 2:

$$\cos\theta_2 = \frac{x_c^2 + y_c^2 - a_2^2 - a_3^2}{2a_2a_3} = D$$

$$\theta_2 = \text{atan2}(D, \sqrt{1-D^2})$$

$$\theta_1 = \text{atan2}(x_c, y_c) - \text{atan2}(a_2 + a_3\cos\theta_2, a_3\sin\theta_2)$$

$$d_3 = a_1 - d_4 - Z_c$$



step 3:

Calculate ${}^0R_3$ based on θ_1, θ_2, d_3

$\theta_1 = \gamma - \beta$

$\gamma = \tan^{-1}\frac{y_c}{x_c}$

$\beta = \tan^{-1}\frac{a_3\sin\theta_2}{a_2 + a_3\cos\theta_2}$

step 4:

${}^3R_4 = [{}^0R_3]^{-1}\, R_{EE}$

${}^3R_4$: is a rotation about Z_3 axis (i.e $-Z_0$)

$${}^3R_4 = \begin{bmatrix} C\theta_4 & -S_4 & 0 \\ S\theta_4 & C_4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \theta_4 = \text{atan2}(r_{11}, r_{21})$$

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